What is a grassmannian?

K - Field (division algebra)

E - vector space over K

```
def Grassmannian K E r :=
  {W : Submodule K E | FiniteDimensional K W ∧ FiniteDimensional.finrank K W = r}
```

ex: $\boxed{r=1}$ Grassmannian K E 1 = { lines in E }

= projective space of E

Another point of view

Mathlib definition of $\mathbb{P}$ K E:

```
Projectivization K E := Quotient (projectivizationSetoid K E)
```

where:

```
projectivizationSetoid K E : Setoid { v : E // v ≠ 0 }:= Setoid.comap Subtype.val (MulAction.orbitRel Kˣ E)
```

In other words, $\mathbb{P}\ K\ E = (E - \{0\}) / v \sim \lambda v$

$\forall v \in E - \{0\},\ \forall \lambda \in K^\times$.

The case of grassmannians:

```
def QGrassmannian K E (Fin r) := Quotient (grassmannianSetoid K E (Fin r))
```

where:

```
def grassmannianSetoid : Setoid { v : I → E // LinearIndependent K v} :=
  Setoid.comap (fun v => Submodule.span K (Set.range v.1)) ⟨(· = ·), eq_equivalence⟩
```

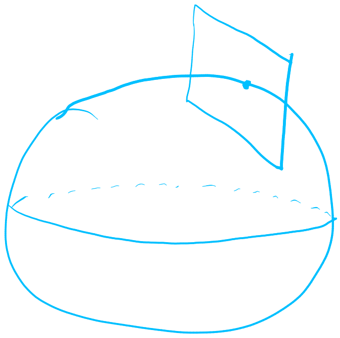
In other words:

* $E^{r\circ} = \{(v_1, \dots, v_r) \in E^r \text{ linearly independent}\}$
* Grassmannian K E r $= E^{r\circ}/\sim$ with

$(v_1, \dots, v_r) \sim (w_1, \dots, w_r)$ iff $\mathrm{Span}(v_i) = \mathrm{Span}(w_i)$.

What is a manifold?

It is a space that looks locally like an open subset of $\mathbb{R}^n$.



More formally, a manifold is:

* a topological space M
* a family of charts (= an atlas)

$$\underset{\substack{\text{open}\\ \text{in } M}}{\varphi_i : U_i} \xrightarrow[\text{homeo}]{\sim} V_i \text{ open in } \mathbb{R}^{n_i}$$

such that: (a) the charts cover M: $M = \bigcup U_i$

(b) change of charts are smooth: $\forall\, i, j$

$$\begin{tikzcd}
\mathbb{R}^{n_i} \supset V_i & U_i \arrow[l, "\varphi_i"', "\sim"] & & U_j \arrow[r, "\varphi_j", "\sim"'] & V_j \subset \mathbb{R}^{n_j} \\
\varphi_i(U_i \cap U_j) \arrow[u, hook, "\text{open}"'] & & U_i \cap U_j \arrow[ul] \arrow[ur] \arrow[ll, "\varphi_i", "\sim"'] \arrow[rr, "\varphi_j"', "\sim"] & & \varphi_j(U_i \cap U_j) \arrow[u, hook, "\text{open}"]
\arrow[from=2-1, to=2-5, dashed, bend right, "\varphi_j \circ \varphi_i^{-1} \text{ is smooth}"']
\end{tikzcd}$$

Mathlib implementation:

(1) The model space is fixed (and more general).

(2) $\forall x \in M$, we give a chart around x.

The grassmannian as a manifold

K - nontrivially valued field (e.g. $\mathbb{R}$)

E - normed K-space (e.g. $\mathbb{R}^k$)

G = Grassmannian K E r

$= \{ W \subset E \;\; K\text{-subspace of dimension } r \}$

$\simeq \underbrace{\{ (v_1, \dots, v_r) \in E^r \text{ linearly independent} \}}_{E^{r0}\text{, open in } E^r} / \sim$

$\rightarrow$ Grassmannian K E r gets the quotient topology

Desiderata: $E^{r0} \xrightarrow[\text{quotient map}]{\pi} G$ = Grassmannian K E r

should be smooth.

Case $r=2$, $E=\mathbb{R}^2$

Elements x of G are represented by $2\times n$ matrices of rank 2

$$V = \begin{pmatrix} a_{11} & a_{12} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix} \circlearrowleft Gl_2(K)$$

$rk(V)=2 \iff V$ has a nonzero 2×2 minor

So: $G = \bigcup_{1\le i<j\le n} U_{ij}$ ← locus where the minor of rows i and j is $\neq 0$

What does U_{12} look like?

Multiplying V on the right by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}$, we get a representative of the same $x\in G$:

$$V' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ \vdots & \vdots \\ * & * \end{pmatrix}$$

$2\times(n-2)$ matrix, arbitrary determined by x

→ this gives a chart

$$\psi_{12}: U_{12} \xrightarrow{\sim} \mathbb{R}^{2(n-2)}$$

Change of chart:

$$U_{12} \cap U_{34}$$

$$V' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ \vdots & \vdots \\ * & * \end{pmatrix} \in \qquad \longmapsto \qquad V' \times \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}^{-1}$$

smooth map of the entries of V'

Coordinate-free version

What do we need to define a chart?

2 "coordinates" on vectors of E

+ the "remainder" of a vector

→ charts are parametrized by

$$\varphi : E \xrightarrow{\sim} K^2 \times U$$

K-linear homeo

normed K-space

$\varphi = (\varphi_1, \varphi_2)$

$\begin{cases} \varphi_1 : E \to K^2 \\ \varphi_2 : E \to U \end{cases}$

Domain of definition of Chart$_\varphi$:

```
def Goodset : Set (Grassmannian K E 2) :=
{W : Grassmannian K E r | W.1 ⊓ (LinearMap.ker φ₁) = ⊥}
```

$\iff \varphi_1|_W$ is injective

$\iff \varphi_1|_W : W \to K^2$ is an isomorphism

Formula for the chart:

$$\mathrm{Chart}_\varphi(W) = \varphi_{2|W} \circ (\varphi_{1|W})^{-1} \quad : \quad K^2 \to U$$

(K-linear continuous)

Inverse of the chart:

If $f : K^2 \to U$ is K-linear continuous,

$$\mathrm{InverseChart}_\varphi(f) = \varphi^{-1}(\mathrm{Graph}(f)).$$

Issues:

* The model space for Chart_φ is (Fin 2 → K) →L[K] U which depends on U.

  What is U? A 2-codimensional subspace of E.
  There is no natural choice, unless E is finite-dimensional.

* We need to choose a chart around each $x \in G$.
  Again, there is no natural choice.

* We need charts to exist!
  Sufficient (and necessary) hypothesis: SeparatingDual K E
  where:

```
class SeparatingDual (R V : Type*) [Ring R] [AddCommGroup V] [TopologicalSpace V]
    [TopologicalSpace R] [Module R V] : Prop :=
  /-- Any nonzero vector can be mapped by a continuous linear map to a nonzero scalar. -/
  exists_ne_zero' : ∀ (x : V), x ≠ 0 → ∃ f : V →L[R] R, f x ≠ 0
```

OK if:
* $K = \mathbb{R}, \mathbb{C}$ ($\mathbb{Q}_p$...) by Hahn-Banach
* $\dim_K E < +\infty$
* E is the dual of a normed space
  (→ "true" grassmannian)

Smooth maps

$$E^{r0} \xrightarrow{\pi} \text{Grassmannian}_K E_r \xrightarrow{f} M$$

manifold

Theorem: f smooth $\Longleftrightarrow$ $f\circ\pi$ smooth.

Applications:

(I) Smooth action of $\mathrm{Aut}(E)$ on Grassmannian$_K E_r$:
(E complete)

$$(g, W) \longmapsto g(W)$$

(II) Plücker embedding:

ex: $E = \mathbb{R}^n$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix} \longmapsto \text{vector of } 2\times 2 \text{ minors,}$$

nonzero element of $\mathbb{R}^{\{(i,j)\,|\,1\le i<j\le n\}}$

defines:
$$\text{Grassmannian}_{\mathbb{R}} E_2 \longrightarrow \mathbb{P}(\underbrace{\mathbb{R}^{n(n-1)/2}}_{\Lambda^2 E})$$

General situation:

$$\text{Grassmannian}_K E_r \xrightarrow{P} \mathbb{P}(\Lambda^r E)$$
$$W \longmapsto \Lambda^r W$$

Issues:

* Exterior powers in mathlib: only the definition exists.
  (need: functoriality, basis, dimension formula...)

* We need a norm on $\Lambda^r E$.

  Several choices, we choose the one that makes the universal property true.

  $\longrightarrow$ need continuous alternating maps

* We need continuous alternating/multilinear maps to be C^∞.

$$\begin{tikzcd}
E^{r,o} \arrow[r, "\text{continuous alternating map}"] \arrow[d, "\pi"'] & \Lambda^r E - \{0\} \arrow[d, "\pi"] \\
\text{Grassmannian } K\,E\,r \arrow[r, "p"'] & \mathbb{P}(\Lambda^r E)
\end{tikzcd}$$

TODO:

* Quaternionic variant.
* Better norm on $\Lambda^r E$. ($\to$ universe issue)
* Compactness of Grassmannian $K\,E\,r$.
* The Plücker map is a homeomorphism on its image + equations of the image.
* Tautological rank r vector bundle on the grassmannian.
  (The fiber at W is W.)
* Universal property. (which one?)
* More smooth maps: Veronese, Segre...
* Algebro-geometric version.